Introduction

Even for smooth data, the optimal state in boundary control problems for the wave equation is in general *discontinuous*. The continuity of the optimal state can be formulated as an *additional requirement*. In Gugat [2006], a boundary control problem for the 1D wave equation is considered, with a constraint that the state is at rest at terminal time T. The continuity of the state is achieved by imposing conditions which require the compatibility of the boundary controls with the initial data. Moreover, the objective function (the sum of the L^2 norms of the time derivatives of the controls) requires the continuity in time of the controls.

This problem and analytical solution appear as [Gugat, 2006, Example 13.2].

Variables & Notation

Unknowns

$$\begin{aligned} u_0 &\in H^1(0,T) && \text{control at the boundary } x = 0 \\ u_L &\in H^1(0,T) && \text{control at the boundary } x = L \\ y &\in H^1((0,L)\times(0,T)) && \text{state variable} \end{aligned}$$

Given Data

The given data is chosen in a way which admits an analytic solution.

$$\begin{aligned} L &= 1 && \text{length of spatial domain } (0,L) \\ T &= 2 && \text{final time} \\ y_0(x) &= -1 && \text{initial elongation at time zero} \\ y_1(x) &= 0 && \text{initial velocity at time zero} \end{aligned}$$

Problem Description

$$
\begin{aligned}
\text{Minimize} \quad & \int_0^T \left[u_0'(s)^2 + u_L'(s)^2\right] \mathrm{d}s \\
\text{s.t.} \quad & \begin{cases}
y_{tt}(x,t) = y_{xx}(x,t) & \text{in } (0,L)\times(0,T) \\
y(x,0) = y_0(x) & \text{in } (0,L) \\
y_t(x,0) = y_1(x) & \text{in } (0,L) \\
y(0,t) = u_0(t) & \text{in } (0,T) \\
y(L,t) = u_L(t) & \text{in } (0,T) \\
y(x,T) = 0 & \text{in } (0,L) \\
y_t(x,T) = 0 & \text{in } (0,L)
\end{cases} \\
\text{and} \quad & \begin{cases}
y(0,0) = u_0(0) \\
y(L,0) = u_L(0) \\
y(0,T) = u_0(T) \\
y(L,T) = u_L(T).
\end{cases}
\end{aligned}
$$

The state is forced to rest at terminal time T. The final block of constraints represents the compatibility conditions which lead to the continuity of the state, together with the continuity of the controls induced by the objective.

Supplementary Material

The optimal controls are given by

$$u_0(t) = u_L(t) = -1 + \frac{t}{2}, \quad t \in [0,T].$$

Figure 0.1 shows two views of the optimal state. The state can be obtained from [Gugat, 2006, Section 11.2.3] as

$$y(x,t) = \frac{1}{2}\left[\alpha(x+t) + \beta(x-t)\right]$$

with the functions

$$\alpha(s) = \begin{cases} -1 & s \in [0,1] \\ s-2 & s \in [1,2] \\ 0 & s \in [2,3] \end{cases}$$

$$\beta(s) = \begin{cases} 0 & s \in [-2,-1] \\ -s-1 & s \in [-1,0] \\ -1 & s \in [0,1]. \end{cases}$$

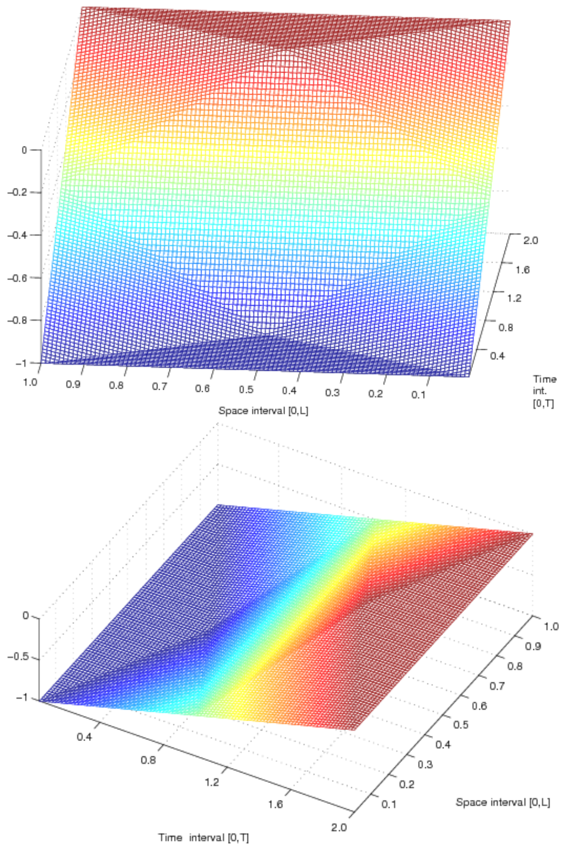

Figure 0.1: Optimal state viewed from different angles as given in [Gugat, 2006, Figure 1(a) and 1(b)]

References

M. Gugat. Optimal boundary control of string to rest in finite time with continuous state. *Zeitschrift für Angewandte Mathematik und Mechanik*, 86:134–150, 2006. doi: 10.1002/zamm.200410236.