## Introduction

This is an exact controllability problem for the one-dimensional wave equation by Dirichlet boundary actuation. Such problems were thorougly analyzed in Gugat et al. [2005], where the present problem appears as Example 1.

## Variables & Notation

### Unknowns

$$\begin{aligned} f_1, f_2 \in L^\infty(0,T) &\quad \text{control variables} \\ y &\quad \text{state variable} \end{aligned}$$

### Given Data

The given data is chosen in a way which admits an analytic solution.

$$\begin{aligned} y_0(x) &= x - 1/2 && \text{target displacement} \\ y_1(x) &= 1 && \text{target velocity} \\ T &= 3.25 && \text{final time} \\ \Omega &= (0,1) && \text{computational domain} \\ \widehat{r} &= 5/28 && \text{auxiliary} \\ g_1(t) &= -1/4 && \text{auxiliary function} \\ g_2(t) &= 3/4 - t && \text{auxiliary function.} \end{aligned}$$

## Problem Description

$$\text{Minimize} \quad \max\left\{\|f_1\|_{L^\infty(0,T)}, \|f_2\|_{L^\infty(0,T)}\right\}$$

$$\text{s.t.} \quad \begin{cases} y_{tt} = y_{xx} & \text{in } \Omega \times (0,T) \\ y(x,0) = 0 & \text{in } \Omega \qquad y_t(x,0) = 0 \quad \text{in } \Omega \\ y(x,T) = y_0(x) & \text{in } \Omega \qquad y_t(x,T) = y_1(x) \quad \text{in } \Omega \\ y(0,t) = f_1(t) & \text{in } (0,T) \qquad y(1,t) = f_2(t) \quad \text{in } (0,T). \end{cases}$$

## Optimality System

An optimality system is not provided in Gugat et al. [2005] but rather the exact solution is constructed analytically.

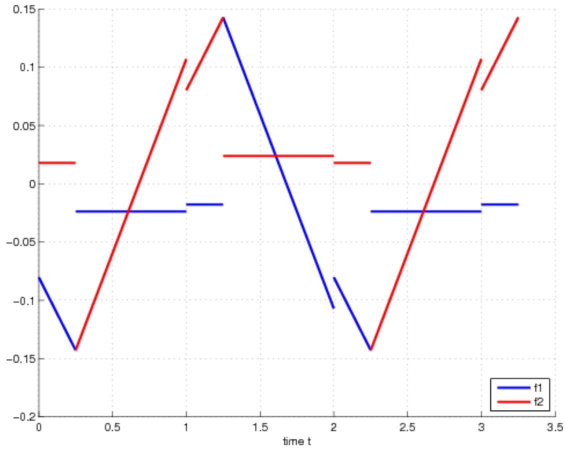


Figure 0.1: optimal controls

## Supplementary Material

An optimal control can be derived analytically from [Gugat et al., 2005, Theorem 2.2]:

$$f_1(T-t) = \begin{cases} \frac{g_1(t)+\widehat{r}}{4}, & t \in [0, 0.25] \\ \frac{g_1(t)+\widehat{r}}{3}, & t \in [0.25, 1] \\ \frac{-g_2(t-1)+\widehat{r}}{4}, & t \in [1, 1.25] \\ \frac{-g_2(t-1)+\widehat{r}}{3}, & t \in [1.25, 2] \\ \frac{g_1(t-2)+\widehat{r}}{4}, & t \in [2, 2.25] \\ \frac{g_1(t-2)+\widehat{r}}{3}, & t \in [2.25, 3] \\ \frac{-g_2(t-3)+\widehat{r}}{4}, & t \in [3, 3.25] \end{cases} \qquad f_2(T-t) = \begin{cases} \frac{g_2(t)-\widehat{r}}{4}, & t \in [0, 0.25] \\ \frac{g_2(t)-\widehat{r}}{3}, & t \in [0.25, 1] \\ \frac{-g_1(t-1)-\widehat{r}}{4}, & t \in [1, 1.25] \\ \frac{-g_1(t-1)-\widehat{r}}{3}, & t \in [1.25, 2] \\ \frac{g_2(t-2)-\widehat{r}}{4}, & t \in [2, 2.25] \\ \frac{g_2(t-2)-\widehat{r}}{3}, & t \in [2.25, 3] \\ \frac{-g_1(t-3)-\widehat{r}}{4}, & t \in [3, 3.25]. \end{cases}$$

This pair of optimal controls may not be unique, but it is the unique one with minimal $L^2(0,T)$ norm. The corresponding displacement $y$ is piecewise linear, and the optimal velocity $y_t$ is piecewise constant on areas bounded by characteristic curves of the equation $y_{tt} = y_{xx}$. Figure 0.1 displays the optimal controls. A plot of the optimal displacement is provided in [Gugat et al., 2005, Figure 2.1].

## References

M. Gugat, G. Leugering, and G. Sklyar. $L^p$-optimal boundary control for the wave equation. *SIAM Journal on Control and Optimization*, 44(1):49–74, 2005. ISSN 0363-0129. doi: 10.1137/S0363012903419212.