Introduction

Here, we have a distributed optimal control problem of the Poisson equation with pointwise box constraints on the control and a one-sided pointwise state constraint. The present problem is given on the unit ball in $\Omega = B_1(0) \subset \mathbb{R}^3$. The control acts in a distributed way on the entire domain Ω and the state constraint is enforced on the entire domain, too. This problem and the analytical solution appear in [Rösch and Steinig, 2012, Section 8]. The problem is designed to have a vanishing Lagrange multiplier for the state constraint. It is thus potentially a good test case for a posteriori error estimation.

Variables & Notation

Unknowns

$$
\begin{aligned}
u &\in L^2(\Omega) && \text{control variable}\\
y &\in H^1(\Omega) && \text{state variable}
\end{aligned}
$$

Given Data

The given data is chosen in a way which admits an analytic solution.

$$
\begin{aligned}
\Omega &= B_1(0) \subset \mathbb{R}^3 && \text{computational domain}\\
&\partial\Omega && \text{its boundary}\\
y_d(x) &= -4\pi^2|x|^2\sin(\pi|x|^2) + 6\pi\cos(\pi|x|^2) + \cos(\frac{\pi}{2}|x|^2) && \text{desired state}\\
f(x) &= 3\pi\sin(\frac{\pi}{2}|x|^2) + \pi^2|x|^2\cos(\frac{\pi}{2}|x|^2) + \sin(\pi|x|^2) && \text{source shift}\\
y_c(x) &= \begin{cases} \cos(\frac{\pi}{2}|x|^2) & \text{if } |x| \le 0.5\\ (-\frac{4}{3}\cos(\frac{\pi}{8}) - \frac{40}{3})|x|^2 + \frac{4}{3}\cos(\frac{\pi}{8}) + \frac{10}{3} & \text{else} \end{cases} && \text{state constraint}
\end{aligned}
$$

Problem Description

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}\\
\text{s.t.} \quad & \begin{cases} -\triangle y = u + f & \text{in } \Omega\\ \quad\;\; y = 0 & \text{on } \partial\Omega \end{cases}\\
\text{and} \quad & -1 \le u(x) \le 1 \quad \text{in } \Omega.\\
\text{and} \quad & y_c(x) \le y(x) \quad \text{in } \Omega
\end{aligned}
$$

Optimality System

The following optimality system, given in the strong form, for the state $y \in H_0^1(\Omega)$, the control $u \in L^2(\Omega)$, the adjoint state $p \in H_0^1(\Omega)$ and the Lagrange multiplier $\mu \in \mathcal{M}(\Omega)$ characterizes the unique minimizer.

$$\begin{aligned}
-\triangle y &= u + f && \text{in } \Omega,\\
y &= 0 && \text{on } \partial\Omega,\\
-\triangle p &= y - y_d - \mu && \text{in } \Omega,\\
p &= 0 && \text{on } \partial\Omega,\\
u &= \operatorname{proj}_{[-1,1]}(-p) && \text{in } \Omega,\\
\langle \mu, y - y_c\rangle_{C(\bar\Omega)^*, C(\bar\Omega)} &= 0, &&\\
\mu &\geq 0, &&\\
y_c &\leq y. &&
\end{aligned}$$

Supplementary Material

The optimal state and control are known analytically:

$$\begin{aligned}
y &= \cos(\frac{\pi}{2}|x|^2),\\
p &= \sin(\pi|x|^2),\\
u &= -\sin(\pi|x|^2),\\
\mu &= 0.
\end{aligned}$$

Note that the state constraint is active on the ball $|x| \leq \frac{1}{2}$. This means that strict complementarity fails on the active set, a fact which makes the problem numerically challenging.

References

A. Rösch and S Steinig. A priori error estimates for a state-constrained elliptic optimal control problem. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46(5): 1107–1120, 2012. doi: 10.1051/m2an/2011076.