Introduction

This example is taken from Günther and Hinze [2008]. It features a complex active set structure for the inequality constraints on the state.

Variables & Notation

Unknowns

$$u \in L^2(\Omega) \quad \text{control variable}$$
$$y \in H^1(\Omega) \quad \text{state variable}$$

Given Data

$$\Omega = (0,1)^2 \qquad \text{computational domain}$$
$$\Gamma \qquad \text{its boundary}$$
$$u_0 = 60 \qquad \text{desired control}$$
$$y_0 = 0.5 \qquad \text{desired state}$$
$$a = 0.45 \qquad \text{lower bound for the state}$$
$$b(x_1, x_2) = \min\left\{1, \max\left\{0.5, 50\left((x_1 - 0.3)^2 + (x_2 - 0.3)^2\right)\right\}\right\} \qquad \text{upper bound for the state}$$

Problem Description

$$\text{Minimize} \quad \frac{1}{2}\|y - y_0\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u - u_0\|_{L^2(\Omega)}^2$$

$$\text{s.t.} \quad \begin{cases} -\triangle y + y = u & \text{in } \Omega \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \Gamma \end{cases}$$

$$\text{and} \quad a \leq y(x) \leq b(x) \quad \text{in } \overline{\Omega}.$$

Optimality System

The following optimality system for the state $y \in H_0^1(\Omega)$, the control $u \in L^2(\Omega)$, the adjoint state $p \in H_0^1(\Omega)$, and the Lagrange multipliers $\mu^a, \mu^b \in \mathcal{M}(\Omega) = C(\overline{\Omega})^*$ for the lower and upper inequality constraint, respectively, given in the strong form, characterizes the unique minimizer.

$$-\triangle y + y = u \qquad \text{in } \Omega,$$

$$\frac{\partial y}{\partial n} = 0 \qquad \text{on } \Gamma,$$

$$-\triangle p + p = y - y_0 + \mu^b - \mu^a \qquad \text{in } \Omega,$$

$$\frac{\partial p}{\partial n} = 0 \qquad \text{on } \Gamma,$$

$$u = u_0 - p,$$

$$\mu^a \geq 0,$$

$$\mu^b \geq 0,$$

$$\int_\Omega (a - y) \, \mathrm{d}\mu^a = 0,$$

$$\int_\Omega (y - b) \, \mathrm{d}\mu^b = 0,$$

$$a \leq y \leq b.$$

Supplementary Material

A reference value for the functional is provided in Günther and Hinze [2008] as

$$J^* \approx 1759.04686$$

References

A. Günther and M. Hinze. A-posteriori error control of a state constrained elliptic control problem. *Journal of Numerical Mathematics*, 16:307–322, 2008. doi: 10.1515/JNUM.2008.014.