Introduction

This problem was introduced in [Benedix and Vexler, 2009, Example 1]. The example is constructed such that the analytic solution in known, and the Lagrange multiplier for the pointwise state constraints is given by a line measure and a volume contribution using smooth problem data. The example is parametrized by three real numbers that allow to steer the contributions of the volume and line measure. Further, the position of the line measure can be chosen to avoid that it coincides with edges of the discretization.

Variables & Notation

Unknowns

$$
\begin{aligned}
q &\in L^2(\Omega) && \text{control variable} \\
u &\in H^1(\Omega) && \text{state variable}
\end{aligned}
$$

Free parameters

Within the problem, there are three free parameters (s, m, b) that can be chosen by the user:

- The boundary between active and inactive set is given by $\{s\} \times [0, 1]$, with the corresponding parameter
$$s \in (0, 1).$$
- To steer the volume contribution, a parameter m can be tuned. Smaller values of m imply a stronger contribution of the line measure. It is assumed that
$$m < s^{-3}.$$
- To steer the volume contribution, a parameter b can be chosen. Larger values of b imply stronger contribution of the volume measure. It is assumed that
$$b > 0.$$

The calculations in Benedix and Vexler [2009] were conducted using $b = 50$, $m = -2$, and $s = 0.125$ as well as $s = 0.3$.

Given Data

The given data is chosen in a way which admits an analytic known solution. The solution is constant along the x_2 direction.

$$\begin{aligned}
&\Omega = (0,1)^2 && \text{computational domain,}\\
&\Gamma && \text{its boundary,}\\
&\Gamma_1 = \{x = (x_1, x_2) \in \Gamma \,|\, x_1 = 0\} && \text{Dirichlet boundary,}\\
&\Gamma_2 = \Gamma \setminus \Gamma_1 && \text{Neumann boundary.}
\end{aligned}$$

With this one defines

$$H^1_D(\Omega) = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma_1\}.$$

The desired state is given by

$$u_d(x_1, x_2) = \begin{cases} x_1^3\, s^{-3} - 3\, x_1^2\, s^{-2} + 3\, x_1\, s^{-1} + 2, & x_1 < s,\\ -\frac{3m}{4(1-s)}(x_1 - s)^4 + m\,(x_1 - s)^3 + 3, & x_1 \geq s. \end{cases}$$

The upper bound is given by

$$u_b(x_1, x_2) = \begin{cases} 1, & x_1 < s,\\ -\frac{3m}{4(1-s)}(x_1 - s)^4 + m\,(x_1 - s)^3 + 1, & x_1 \geq s. \end{cases}$$

Moreover, the uncontrolled force is set to be

$$f(x_1, x_2) = \begin{cases} 6\, s^{-2} - 6\, m\, x_1 + x_1\,(x_1 - 2) + b\,(1 - s)\, x_1, & x_1 < s,\\ (1 - r)\, x_1^2 + (b - \frac{18ms}{1-s} - 2 - 6\, m)\, x_1 + 6\, s^{-2} - r\, s^2, & x_1 \geq s, \end{cases}$$

where

$$r = \frac{b}{2} - \frac{9\, m}{1 - s}.$$

Problem Description

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|q\|^2_{L^2(\Omega)}\\
&\text{s.t.} \quad \begin{cases} -\triangle u = q + f & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma_1,\\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2, \end{cases}\\
&\text{and} \quad u(x) \leq u_b \quad \text{in } \overline{\Omega}.
\end{aligned}$$

Optimality System

The following optimality system for the state $u \in H^1_D(\Omega)$, the control $q \in L^2(\Omega)$, the adjoint state $z \in H^1_D(\Omega)$, and the Lagrange multiplier for the inequality constraints $\mu \in \mathcal{M}(\Omega) = C(\overline{\Omega})^*$ given in the strong form, characterizes the unique minimizer.

$$
\begin{aligned}
-\triangle u &= q + f && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_2,\\[1em]
-\triangle z &= u - u_d + \mu && \text{in } \Omega,\\
z &= 0 && \text{on } \Gamma_1,\\
\frac{\partial z}{\partial n} &= 0 && \text{on } \Gamma_2,\\[1em]
q &= -z && \text{in } \Omega,\\[1em]
\mu &\geq 0 && \text{in } \mathcal{M}(\Omega),\\
\int_\Omega (u - u_b)\,\mathrm{d}\mu &= 0,\\
u &\leq u_b && \text{on } \overline{\Omega}.
\end{aligned}
$$

Supplementary Material

The optimal solution is known analytically. It is given by

$$
\begin{aligned}
u(x_1,x_2) &= \begin{cases} x_1^3\,s^{-3} - 3\,x_1^2\,s^{-2} + 3\,x_1\,s^{-1}, & x_1 < s,\\ u_b(x_1,x_2), & x_1 \geq s,\end{cases}\\
q(x_1,x_2) &= \begin{cases} -x_1(x_1-2) - (6\,s^{-3} - 6\,m)\,x_1 - b\,(1-s)\,x_1, & x_1 < s,\\ -x_1(x_1-2) - 6\,s^{-2} + 6\,m\,s + \frac{b}{2}\,x_1^2 - b\,x_1 + \frac{b}{2}s^2, & x_1 \geq s,\end{cases}\\
z(x_1,x_2) &= -q(x_1,x_2)\\
\mu = \mu_1 &+ \mu_2\\
\int_\Omega \phi\,\mathrm{d}\mu_1 &= \left(\frac{6}{s^3} - 6\,m\right)\int_0^1 \phi(s,x_2)\,\mathrm{d}x_2 \quad \forall \phi \in C(\overline{\Omega}),\\
\mu_2(x_1,x_2) &= \begin{cases} 0, & x_1 < s,\\ b, & x_1 \geq s.\end{cases}
\end{aligned}
$$

References

O. Benedix and B. Vexler. A posteriori error estimation and adaptivity for elliptic optimal control problems with state constraints. *Computational Optimization and Applications*, 44(1):3–25, 2009. doi: 10.1007/s10589-008-9200-y.