Introduction

This problem is a standard linear-quadratic optimal control problem with pointwise state constraints, but no constraints on the control variable. It appears in Meyer et al. [2005] as a test example for the so-called Lavrentiev regularization approach. By the latter, optimal control problems with state constraints of the form $y(x) \ge y_c(x)$ are approximated by problems involving mixed control-state constraints $y(x)+\varepsilon\, u(x) \ge y_c(x)$. While problems with pointwise state constraints usually involve a measure-valued Lagrange multiplier, see Casas [1986], the corresponding quantity in the regularized problem is a function.

The following state-constrained problem and its solution appear in [Meyer et al., 2005, Section 7, Example 2] and it features a line measure as the state constraint multiplier.

Variables & Notation

Unknowns

$$u \in L^2(\Omega) \quad \text{control variable}$$
$$y \in H^1(\Omega) \quad \text{state variable}$$

Given Data

The given data is chosen in a way which admits an analytic solution.

$$\Omega = (0,1)^2 \quad \text{computational domain}$$
$$\Omega_1 = \{x = (x_1, x_2) \in \Omega : x_1 < 0.5\} \quad \text{subdomain of } \Omega$$
$$\Omega_2 = \{x = (x_1, x_2) \in \Omega : x_1 > 0.5\} \quad \text{subdomain of } \Omega$$
$$\widehat{\Gamma} = \{x = (x_1, x_2) \in \Omega : x_1 = 0.5\} \quad \text{line inside } \Omega$$
$$y_d = \begin{cases} -0.5 + x_1^2, & x_1 < 0.5 \\ 0.75, & x_1 \ge 0.5 \end{cases} \quad \text{desired state (discontinuous)}$$
$$u_d = \begin{cases} 2.5 - x_1^2, & x_1 < 0.5 \\ 2.25, & x_1 \ge 0.5 \end{cases} \quad \text{desired control (continuous)}$$
$$y_c = \begin{cases} 2\,x_1 + 1, & x_1 < 0.5 \\ 2, & x_1 \ge 0.5 \end{cases} \quad \text{state constraint (continuous)}$$

Note that there is a typo in the specification of y_d in [Meyer et al., 2005, Section 7, Example 2].

Problem Description

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} \\
\text{s.t.} \quad & \begin{cases} -\triangle y + y = u & \text{in } \Omega \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \\
\text{and} \quad & y \geq y_c \quad \text{in } \overline{\Omega}.
\end{aligned}$$

Optimality System

The following optimality system for the state $y \in H^1(\Omega)$, the control $u \in L^2(\Omega)$, the adjoint state $p \in H^1(\Omega)$ and the state constraint multiplier $\mu \in M(\overline{\Omega})$, given in the strong form, characterizes the unique minimizer.

$$\begin{aligned}
-\triangle y + y &= u && \text{in } \Omega \\
\frac{\partial y}{\partial n} &= 0 && \text{on } \partial\Omega \\
-\triangle p + p &= y - y_d - \mu_\Omega && \text{in } \Omega \\
\frac{\partial p}{\partial n} &= -\mu_{\partial\Omega} && \text{on } \partial\Omega \\
u - u_d + p &= 0 && \text{in } \Omega \\
y - y_c &\geq 0 && \text{in } \overline{\Omega} \\
\mu &\geq 0 && \text{in } M(\overline{\Omega}) \\
\langle \mu, y - y_c \rangle_{M(\overline{\Omega}),C(\overline{\Omega})} &= 0
\end{aligned}$$

The space $M(\overline{\Omega})$ is the dual of $C(\overline{\Omega})$ and it consists of all signed, real, regular Borel measures on $\overline{\Omega}$. Here, μ_Ω and $\mu_{\partial\Omega}$ denote the restrictions of the measure μ to Ω and its boundary, respectively.

Supplementary Material

The optimal state, adjoint state, control and state constraint multiplier are known analytically:

$$y = 2$$

$$p = \begin{cases} 0.5 - x_1^2, & x_1 < 0.5 \\ 0.25, & x_1 \geq 0.5 \end{cases}$$

$$u = 2$$

$$\langle \mu, y \rangle_{M(\overline{\Omega}),C(\overline{\Omega})} = \int_{\Omega_2} y(x)\,\mathrm{d}x + \int_{\widehat{\Gamma}} y(s)\,\mathrm{d}s \quad \text{for } y \in C(\overline{\Omega}).$$

Note that μ consists of a regular part (the characteristic function of Ω_2) plus a singular part (the line measure concentrated on $\widehat{\Gamma}$). In particular, the boundary part $\mu_{\partial\Omega}$ vanishes.

Revision History

- 2014–11–14: added missing term u_d to the objective (thanks to Stefan Takacs)
- 2012–11–14: problem added to the collection

References

E. Casas. Control of an elliptic problem with pointwise state constraints. *SIAM Journal on Control and Optimization*, 24(6):1309–1318, 1986. doi: 10.1137/0324078.

C. Meyer, A. Rösch, and F. Tröltzsch. Optimal control of PDEs with regularized pointwise state constraints. *Computational Optimization and Applications*, 33(2–3):209–228, 2005. doi: 10.1007/s10589-005-3056-1.