Introduction

We consider the optimal control of static elastoplasticity with linear kinematic hardening. This leads to an optimal control problem governed by an elliptic variational inequality of first kind in mixed form or, equivalently, an MPCC in function space.
The objective functional tracks the state both on the entire domain and additionally on a submanifold of the domain. The control cost is taken into account by a tracking type term as well. There are no constraints on the control, which acts in a distributed way on the domain. The Dirichlet boundary is the entire boundary of the domain.
A locally optimal control is known, whose corresponding state has a bi-active set with a positive measure.

The problem and its solution are taken from [Betz et al., 2014, Section 6.1].

Variables & Notation

Unknowns

$$\boldsymbol{f} \in L^2\big(\Omega;\mathbb{R}^2\big) \qquad \text{control variable}$$

$$\big(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{u},\lambda\big) \in L^2\big(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}}\big)^2 \times H^1_0\big(\Omega;\mathbb{R}^2\big) \times L^2(\Omega) \qquad \text{state variable}$$

The state is composed of the stress $\boldsymbol{\sigma}$, back stress $\boldsymbol{\chi}$, displacement $\boldsymbol{u}$ and plastic multiplier λ.

Given Data

$\Omega = \{x \in \mathbb{R}^2 : \|x\| < 1\}$	computational domain
$B = \{x \in \mathbb{R}^2 : \|x\| < 1/2\}$	subdomain
$R = \Omega \setminus B$	subdomain
$k_1 = 1.0$	hardening constant
$E = 1.0$	elasticity modulus
$\nu = 0.0$	Poisson number
$\mu_L = 0.5$	Lamé constant
$\lambda_L = 0.0$	Lamé constant
$\sigma_0 > 0$	yield stress (arbitrary)
$\alpha > 0$	control cost parameter (arbitrary)

desired displacement in the domain:

$$\boldsymbol{u}_\Omega(x) = \begin{cases} \begin{pmatrix} U\big(\|x\|^2\big) + \big(8x_1^2 + 4x_2^2 + 4x_1x_2\big)U''\big(\|x\|^2\big) + 6U'\big(\|x\|^2\big) \\ U\big(\|x\|^2\big) + \big(4x_1^2 + 8x_2^2 + 4x_1x_2\big)U''\big(\|x\|^2\big) + 6U'\big(\|x\|^2\big) \end{pmatrix}, & x \in B \\ \begin{pmatrix} U\big(\|x\|^2\big) + \big(6x_1^2 + 2x_2^2 + 4x_1x_2\big)U''\big(\|x\|^2\big) + 4U'\big(\|x\|^2\big) \\ U\big(\|x\|^2\big) + \big(2x_1^2 + 6x_2^2 + 4x_1x_2\big)U''\big(\|x\|^2\big) + 4U'\big(\|x\|^2\big) \end{pmatrix}, & x \in R \end{cases}$$

with

$$U(t) = \begin{cases} -\sigma_0\, t^2 + \frac{3}{2}\sigma_0\, t - \frac{13}{16}\sigma_0, & t < \frac{1}{4} \\ \sigma_0 \sqrt{t} - \sigma_0, & t \geq \frac{1}{4} \end{cases}$$

$$U'(t) = \begin{cases} -2\,\sigma_0 t + \frac{3}{2}\sigma_0, & t < \frac{1}{4} \\ 0.5\,\sigma_0\, t^{-1/2}, & t \geq \frac{1}{4} \end{cases}$$

$$U''(t) = \begin{cases} -2\,\sigma_0, & t < \frac{1}{4} \\ -0.25\,\sigma_0\, t^{-3/2}, & t \geq \frac{1}{4} \end{cases}$$

desired displacement on the submanifold ∂B:

$$\boldsymbol{u}_{\partial B}(x) = \begin{pmatrix} -\sigma_0 \\ -\sigma_0 \end{pmatrix}$$

desired control:

$$\boldsymbol{f}_\Omega(x) = \begin{pmatrix} \frac{2}{\alpha}U\big(\|x\|^2\big) - \big(4x_1^2 + 2x_2^2 + 2x_1x_2\big)U''\big(\|x\|^2\big) - 3U'\big(\|x\|^2\big) \\ \frac{2}{\alpha}U\big(\|x\|^2\big) - \big(2x_1^2 + 4x_2^2 + 2x_1x_2\big)U''\big(\|x\|^2\big) - 3U'\big(\|x\|^2\big) \end{pmatrix}$$

Problem Description

$$\text{Minimize} \quad J(\boldsymbol{u}, \boldsymbol{f}) := \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_\Omega\|^2_{L^2(\Omega;\mathbb{R}^2)} + \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_{\partial B}\|^2_{L^2(\partial B;\mathbb{R}^2)} + \frac{\alpha}{2}\|\boldsymbol{f} - \boldsymbol{f}_\Omega\|^2_{L^2(\Omega;\mathbb{R}^2)}$$

$$\text{s.t.} \quad \begin{cases} \mathbb{C}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) = 0 & \text{in } L^2\big(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}}\big), \\ \mathbb{H}^{-1}\boldsymbol{\chi} + \lambda\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) = 0 & \text{in } L^2\big(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}}\big), \\ -\big(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{L^2\left(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}}\right)} + (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^2)} = 0 & \forall\, \boldsymbol{v} \in H^1_0\big(\Omega; \mathbb{R}^2\big), \\ 0 \leq \lambda \perp \phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leq 0 & \text{a.e. in } \Omega. \end{cases}$$

With the given Lamé coefficients, the inverse elasticity and hardening tensors read

$$\mathbb{C}^{-1}\boldsymbol{\sigma} = \frac{1}{2\mu_L}\boldsymbol{\sigma} - \frac{\lambda_L}{2\mu_L(2\mu_L + 2\lambda_L)}\,\text{trace}(\boldsymbol{\sigma})\boldsymbol{I} = \boldsymbol{\sigma},$$

$$\mathbb{H}^{-1}\boldsymbol{\chi} = \frac{1}{k_1}\boldsymbol{\chi} = \boldsymbol{\chi},$$

where $\boldsymbol{I}\colon \mathbb{R}^{2\times 2}_{\text{sym}} \to \mathbb{R}^{2\times 2}_{\text{sym}}$ is the identity mapping and

$$\begin{aligned} \boldsymbol{\varepsilon}(\boldsymbol{u}) &= \frac{1}{2}\big(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\big) && \text{the linearized strain,} \\ \boldsymbol{\sigma}^D &= \boldsymbol{\sigma} - \operatorname{trace}(\boldsymbol{\sigma})\boldsymbol{I} && \text{the deviatoric part of } \boldsymbol{\sigma}. \end{aligned}$$

The yield function is given by

$$\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = \frac{\|\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\|_F^2 - \sigma_0^2}{2},$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. The corresponding inner product $\operatorname{trace}(\boldsymbol{A}^\top \boldsymbol{B})$ is used in the calculation of $(\cdot,\cdot)_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})}$.

Optimality System

According to [Betz and Meyer, 2015, Theorem 4.6] (Theorem 4.4 in the Preprint) the following conditions are sufficient for local optimality of the control $\boldsymbol{f}$ with associated state $(\boldsymbol{\sigma}, \boldsymbol{\chi}, \boldsymbol{u}, \lambda)$.
There exist adjoint variables $(\boldsymbol{\zeta}, \boldsymbol{\psi}, \boldsymbol{w}) \in L^\infty(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})^2 \times H^1_0(\Omega; \mathbb{R}^2)$ and multipliers $(\mu, \theta) \in L^\infty(\Omega) \times L^\infty(\Omega)$ satisfying

1. the optimality system

$$\begin{aligned} \mathbb{C}^{-1}\boldsymbol{\zeta} - \boldsymbol{\varepsilon}(\boldsymbol{w}) + \lambda(\boldsymbol{\zeta}^D + \boldsymbol{\psi}^D) + \theta(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D) &= 0 && \text{in } L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}}) \\ \mathbb{H}^{-1}\boldsymbol{\psi} + \lambda(\boldsymbol{\zeta}^D + \boldsymbol{\psi}^D) + \theta(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D) &= 0 && \text{in } L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}}) \\ -\big(\boldsymbol{\zeta}, \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})} + (\boldsymbol{u} - \boldsymbol{u}_\Omega, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^2)} & \\ + (\boldsymbol{u} - \boldsymbol{u}_{\partial B}, \boldsymbol{v})_{L^2(\partial B;\mathbb{R}^2)} &= 0 && \forall\, \boldsymbol{v} \in H^1_0(\Omega; \mathbb{R}^2) \\ \boldsymbol{w} + \alpha(\boldsymbol{f} - \boldsymbol{f}_\Omega) &= 0 && \text{in } L^2(\Omega; \mathbb{R}^2) \\ (\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D) : (\boldsymbol{\zeta}^D + \boldsymbol{\psi}^D) - \mu &= 0 && \text{a.e. in } \Omega \\ \mu\,\lambda &= 0 && \text{a.e. in } \Omega \\ \theta\,\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) &= 0 && \text{a.e. in } \Omega \\ \mu &\ge 0 && \text{a.e. in } \Omega \\ \theta &\ge 0 && \text{a.e. in } \Omega \end{aligned}$$

2. the second-order condition
   There exists $\kappa > 0$ such that

$$\partial^2_{(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{\chi}, \boldsymbol{u}, \lambda, \boldsymbol{f}, \boldsymbol{\zeta}, \boldsymbol{\psi}, \boldsymbol{w}, \mu, \theta)\,(\boldsymbol{\sigma}', \boldsymbol{\chi}', \boldsymbol{u}', \lambda', \boldsymbol{h})^2 \ge \kappa \|\boldsymbol{h}\|_U^2$$

holds for all $\boldsymbol{h} \in L^2\big(\Omega;\mathbb{R}^2\big)$ and $\big(\boldsymbol{\sigma}',\boldsymbol{\chi}',\boldsymbol{u}',\lambda'\big)$ solving

$$\begin{aligned}
\mathbb{C}^{-1}\boldsymbol{\sigma}' - \boldsymbol{\varepsilon}(\boldsymbol{u}') + \lambda\big((\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\big) + \lambda'\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) &= 0 \quad \text{in } L^2\big(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}}\big)\\
\mathbb{H}^{-1}\boldsymbol{\chi}' + \lambda\big((\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\big) + \lambda'\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) &= 0 \quad \text{in } L^2\big(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}}\big)\\
-\big(\boldsymbol{\sigma}',\boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})} + (\boldsymbol{h},\boldsymbol{v})_{L^2(\Omega;\mathbb{R}^2)} &= 0 \quad \forall\, \boldsymbol{v} \in H^1_0\big(\Omega;\mathbb{R}^2\big)\\
\mathbb{R} \ni \lambda' \perp \big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) : \big((\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\big) &= 0 \quad \text{a.e. in } \mathcal{A}_s\\
0 \le \lambda' \perp \big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) : \big((\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\big) &\le 0 \quad \text{a.e. in } \mathcal{B}\\
0 = \lambda' \perp \big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big) : \big((\boldsymbol{\sigma}')^D + (\boldsymbol{\chi}')^D\big) &\in \mathbb{R} \quad \text{a.e. in } \mathcal{I}.
\end{aligned}$$

The sets are defined as follows,

$$\begin{aligned}
\mathcal{A}_s &:= \{x \in \Omega : \lambda > 0\} && \text{strongly active set,}\\
\mathcal{B} &:= \{x \in \Omega : \phi(\boldsymbol{\sigma},\boldsymbol{\chi}) = \lambda = 0\} && \text{bi-active set,}\\
\mathcal{I} &:= \{x \in \Omega : \phi(\boldsymbol{\sigma},\boldsymbol{\chi}) < 0\} && \text{inactive (elastic) set,}
\end{aligned}$$

and the Lagrangian $\mathcal{L}$ is defined by

$$\begin{aligned}
&\mathcal{L}(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{u},\lambda,\boldsymbol{f},\boldsymbol{\zeta},\boldsymbol{\psi},\boldsymbol{w},\mu,\theta)\\
&\quad = J(\boldsymbol{u},\boldsymbol{f}) + \big(\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big),\boldsymbol{\zeta}\big)_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})}\\
&\qquad + \big(\boldsymbol{\chi} + \lambda\big(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D\big),\boldsymbol{\psi}\big)_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})} - \big(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{w})\big)_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})}\\
&\qquad + (\boldsymbol{f},\boldsymbol{w})_{L^2(\Omega;\mathbb{R}^2)} - (\lambda,\mu)_{L^2(\Omega)} + (\phi(\boldsymbol{\sigma},\boldsymbol{\chi}),\theta)_{L^2(\Omega)}.
\end{aligned}$$

Consequently $\partial^2_{(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{u},\lambda,\boldsymbol{f},\boldsymbol{\zeta},\boldsymbol{\psi},\boldsymbol{w},\mu,\theta\big)\big(\boldsymbol{\sigma}',\boldsymbol{\chi}',\boldsymbol{u}',\lambda',\boldsymbol{h}\big)^2$ is given as follows:

$$\begin{aligned}
&\partial^2_{(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{u},\lambda,\boldsymbol{f})}\mathcal{L}\big(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{u},\lambda,\boldsymbol{f},\boldsymbol{\zeta},\boldsymbol{\psi},\boldsymbol{w},\mu,\theta\big)\big(\boldsymbol{\sigma}',\boldsymbol{\chi}',\boldsymbol{u}',\lambda',\boldsymbol{h}\big)^2\\
&\qquad = (\boldsymbol{u}',\boldsymbol{u}')_{L^2(\Omega;\mathbb{R}^2)} + (\boldsymbol{u}',\boldsymbol{u}')_{L^2(\partial B;\mathbb{R}^2)} + \alpha(\boldsymbol{h},\boldsymbol{h})_{L^2(\Omega;\mathbb{R}^2)}\\
&\qquad\quad + 2\Big(\lambda'\big((\sigma')^D + (\chi')^D\big),\, \zeta^D + \psi^D\Big)_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})}\\
&\qquad\quad + \Big(\big\|(\sigma')^D + (\chi')^D\big\|_F^2,\, \theta\Big)_{L^2(\Omega)}.
\end{aligned}$$

Supplementary Material

Locally optimal control, state, adjoint state and multipliers are known analytically:

$$\boldsymbol{u} = \begin{pmatrix} U\left(\|x\|^2\right) \\ U\left(\|x\|^2\right) \end{pmatrix} \qquad \text{displacement,}$$

$$\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = U'(\|x\|^2) \begin{pmatrix} 2\,x_1 & x_1 + x_2 \\ x_1 + x_2 & 2\,x_2 \end{pmatrix} \qquad \text{stress,}$$

$$\boldsymbol{\chi} = \mathbf{0} \qquad \text{back stress,}$$

$$\lambda = 0 \qquad \text{plastic multiplier,}$$

$$\begin{aligned} \boldsymbol{f} &= -\operatorname{div}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \\ &= -\begin{pmatrix} 3U'(\|x\|^2) + U''(\|x\|^2)(4x_1^2 + 2x_1x_2 + 2x_2^2) \\ 3U'(\|x\|^2) + U''(\|x\|^2)(2x_1^2 + 2x_1x_2 + 4x_2^2) \end{pmatrix} \end{aligned} \qquad \text{control (right hand side force),}$$

$$\boldsymbol{\zeta} = \begin{cases} 2\,\boldsymbol{\varepsilon}(\boldsymbol{u}), & x \in B \\ \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}(\boldsymbol{u})^S, & x \in R \end{cases} \qquad \text{adjoint stress,}$$

$$\boldsymbol{\psi} = \begin{cases} \mathbf{0}, & x \in B \\ -\boldsymbol{\varepsilon}(\boldsymbol{u})^D, & x \in R \end{cases} \qquad \text{adjoint back stress,}$$

$$\boldsymbol{w} = 2\boldsymbol{u} \qquad \text{adjoint displacement,}$$

$$\mu = \begin{cases} 2\,\|\boldsymbol{\varepsilon}(\boldsymbol{u})^D\|_F^2, & x \in B \\ \mathbf{0}, & x \in R \end{cases} \qquad \text{multiplier,}$$

$$\theta = \begin{cases} 0, & x \in B \\ 1, & x \in R \end{cases} \qquad \text{multiplier.}$$

The magnitude of the optimal state and control for the values $\alpha = 1$ and $\sigma_0 = 2$ as given in Betz et al. [2014] are depicted in Figure 0.1. The reference value for the objective,

$$J \approx 156.448\,738\,479\,265\,980\,83$$

corresponding to these values of α and σ_0 is given in Betz et al. [2014].

Revision History

- 2015–04–22: Updated reference to the published version.
- 2014–11–29: Explicit formulas for the second-order derivative of the Lagrangian added. Formulas for optimal $\boldsymbol{\sigma}$ and $\boldsymbol{f}$ added. Pictures of the optimal solution added. Reference for objective value added. Matlab code added.
- 2014–04–23: problem added to the collection

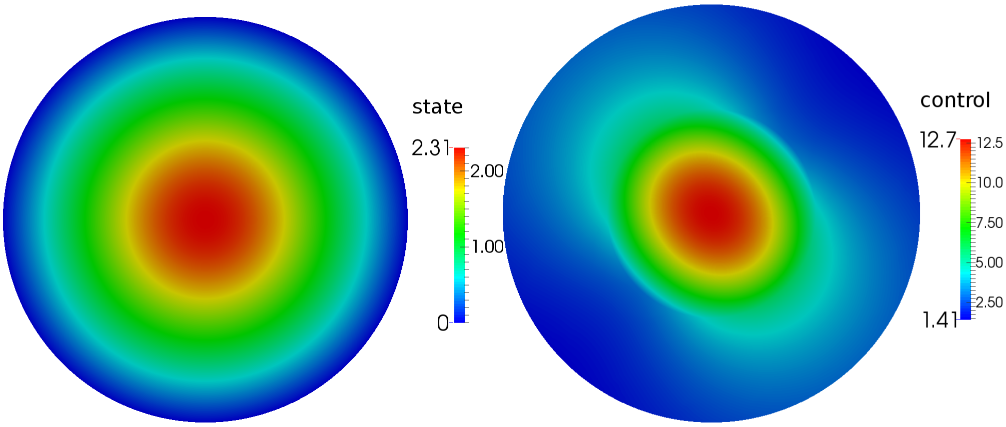

Figure 0.1: Analytical values of the optimal state $|\boldsymbol{u}|$ (left) and control $|\boldsymbol{f}|$ (right) for parameters $\alpha = 1$ and $\sigma_0 = 2$. Figure courtesy of K. Rosin

References

T. Betz and C. Meyer. Second-order sufficient optimality conditions for optimal control of static elastoplasticity with hardening. *ESAIM: Control, Optimisation and Calculus of Variations*, 21(1):271–300, 2015. doi: 10.1051/cocv/2014024.

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