Introduction

This is an optimal control problem for entropy solutions of the inviscid one-dimensional Burgers equation considered in Castro et al. [2008]. The control acts as initial data and the objective function is a tracking type functional at end time with discontinuous desired state.

Variables & Notation

Unknowns

$$\begin{aligned} u = (u_0, u_\ell, u_r) &\in L^\infty(\Omega) \times \mathbb{R} \times \mathbb{R} && \text{control variable} \\ y &\in L^\infty(\mathbb{R} \times (0,T)) && \text{state variable} \end{aligned}$$

Given Data

$$\begin{aligned} &\Omega = (-4, 4) && \text{computational domain} \\ &\Omega_\ell = (-\infty, -4] && \text{outer left domain} \\ &\Omega_\ell = [4, \infty) && \text{outer right domain} \\ &T = 1 && \text{final time} \\ &y_\Omega = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases} && \text{desired state} \end{aligned}$$

Problem Description

$$\begin{aligned} \text{Minimize} \quad & \|y(\cdot,T) - y_\Omega\|^2_{L^2(\Omega)} \\ \text{s.t.} \quad & y \text{ is a weak entropy solution of} \\ & \begin{cases} \dfrac{\partial}{\partial t} y + \dfrac{\partial}{\partial x}\left(\dfrac{y^2}{2}\right) = 0 & \text{in } \mathbb{R} \times (0,T) \\ y(x,0) = \begin{cases} u_\ell & \text{if } x \in \Omega_\ell \\ u_0(x) & \text{if } x \in \Omega \\ u_r & \text{if } x \in \Omega_r \end{cases} & \text{on } \mathbb{R}. \end{cases} \end{aligned}$$

This problem appears as [Castro et al., 2008, Section 7, Experiment 1]. Suitable numerical schemes for the Burgers equation are given in [Castro et al., 2008, Section 3]. Note that the state is expected to have shock discontinuities.

Instead of formulating the Burgers equation on $\mathbb{R} \times (0,T)$, the equation can be restricted to $\Omega \times (0,T)$, and then u_ℓ and u_r can be described as constant boundary data at $x \in \{-4,4\}$. This leads to a very similar problem with the same optimal solution.

Supplementary Material

A globally optimal state and control are known analytically:

$$\begin{aligned} y &= \begin{cases} 1, & x < \frac{t-1}{2}, \\ 0, & x \geq \frac{t-1}{2}, \end{cases} \\ u_0 &= \begin{cases} 1, & x < -\frac{1}{2}, \\ 0, & x \geq -\frac{1}{2}, \end{cases} \\ u_\ell &= 1, \\ u_r &= 0. \end{aligned}$$

Note that the value of the objective is zero for this solution. The optimal state contains a shock which moves through the domain but does not reach the boundary of Ω within the time interval $(0,1)$.

The performance of a gradient descent method and an alternating descent method in combination with various discretizations of the state equation is described in [Castro et al., 2008, Section 7]. The authors use

$$u_0 = \begin{cases} 2, & x < \frac{1}{4}, \\ 0, & x \geq \frac{1}{4}, \end{cases} \qquad u_\ell = 2, \qquad u_r = 0$$

as an initial guess for the control variables as their Experiment 1 and the alternative initial guess

$$u_0 = -1, \qquad u_\ell = -1, \qquad u_r = -1.$$

as their Experiment 2.

References

C. Castro, F. Palacios, and E. Zuazua. An alternating descent method for the optimal control of the inviscid Burgers equation in the presence of shocks. *Mathematical Models & Methods in Applied Sciences*, 18(3):369–416, 2008. doi: 10.1142/S0218202508002723.