Introduction

This is a variation of the *mother problem* with additional pointwise constraints on the gradient of the state with known analytic solution. The presented problem is given on a domain $\Omega \subset \mathbb{R}^2$. This problem and analytical solution where proposed in [Deckelnick et al., 2008, Section 5], and have been verified in Wollner [2010]. The solution of the problem is special due to the fact that no additional bounds on the control are needed.

Variables & Notation

Unknowns

$$\begin{array}{rl} u \in L^2(\Omega) & \text{control variable} \\ y \in H^1(\Omega) & \text{state variable} \end{array}$$

Given Data

The given data is chosen in a way which admits an analytic solution, that is given by rotation of a one dimensional problem.

$$\begin{array}{ll}
\Omega = B_2(0) = \{x \in \Omega : |x| \le 2\} & \text{computational domain} \\
\Gamma & \text{its boundary} \\
y_\Omega(x) = \begin{cases} \frac{1}{4} + \frac{1}{2}\ln 2 - \frac{1}{4}|x|^2, & 0 \le |x| \le 1, \\ \frac{1}{2}\ln 2 - \frac{1}{2}\ln|x|, & 1 < |x| \le 2. \end{cases} & \text{desired state} \\
e_\Omega(x) = \begin{cases} 2, & 0 \le |x| \le 1, \\ 0, & 1 < |x| \le 2. \end{cases} & \text{given right hand side}
\end{array}$$

Problem Description

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{2}\|y - y_\Omega\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} \\
\text{s.t.} \quad & \begin{cases} -\triangle y = u + e_\Omega & \text{in } \Omega \\ y = 0 & \text{on } \Gamma \end{cases} \\
\text{and} \quad & |\nabla y| \le \frac{1}{2} \quad \text{in } \overline{\Omega}.
\end{aligned}$$

Optimality System

The following optimality system for the state $y \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ with $p > 2$, the control $u \in L^2(\Omega)$, the adjoint state $p \in L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$, and a Lagrange multiplier $\mu \in M(\Omega)^2 = C^*(\overline{\Omega})^2$ for the constraint on the gradient of y characterizes the unique minimizer, see Casas and Fernández [1993]:

$$\begin{aligned}
-\triangle y &= u + e_\Omega && \text{in } \Omega\\
y &= 0 && \text{on } \Gamma\\
-\triangle p &= y - y_\Omega + \nabla^*\mu && \text{in } \Omega\\
p &= 0 && \text{on } \Gamma\\
u &= -p && \text{in } \Omega\\
\langle \phi - \nabla y, \mu\rangle_{C,C^*} &\le 0 && \forall \phi \in C(\overline{\Omega})^2, |\phi| \le \frac{1}{2},\\
\langle |\nabla y| - \tfrac{1}{2}, \mu\rangle_{C,C^*} &= 0.
\end{aligned}$$

Here the adjoint equation has to be understood in the very weak sense, i.e., p solves

$$-\int_\Omega p \triangle \phi \,\mathrm{d}x = \int_\Omega (y - y_\Omega)\phi \,\mathrm{d}x + \int_\Omega \nabla\phi \,\mathrm{d}\mu \quad \forall \phi \in H_0^1(\Omega) \cap C^1(\overline{\Omega}).$$

Supplementary Material

The optimal state, adjoint state, control and Lagrange multiplier are known analytically:

$$\begin{aligned}
y &= y_\Omega,\\
p &= -u,\\
u &= \begin{cases} -1, & 0 \le |x| \le 1,\\ 0, & 1 < |x| \le 2,\end{cases}\\
\mu &= \frac{\nabla y}{|\nabla y|}\mu_0,\\
\langle \phi, \mu_0\rangle_{C,C^*} &= \int_{|x|=1} \phi \,\mathrm{d}s.
\end{aligned}$$

References

E. Casas and L. A. Fernández. Optimal control of semilinear elliptic equations with pointwise constraints on the gradient of the state. *Applied Mathematics and Optimization*, 27:35–56, 1993. doi: 10.1007/BF01182597.

K. Deckelnick, A. Günther, and M. Hinze. Finite element approximation of elliptic control problems with constraints on the gradient. *Numerische Mathematik*, 111:335–350, 2008. doi: 10.1007/s00211-008-0185-3.

W. Wollner. A posteriori error estimates for a finite element discretization of interior point methods for an elliptic optimization problem with state constraints. *Computational Optimization and Applications*, 47(1):133–159, 2010. doi: 10.1007/s10589-008-9209-2.