Introduction

This problem is a prototypical example for a convection dominated stationary diffusion problem. Such problems are known to produce boundary layers and to lead to unphysical oscillatory solutions in conventional discretization schemes.

The present problem was originally proposed in [Collis and Heinkenschloss, 2002, Example 3] and discretized using an SUPG (streamline upwind Petrov Galerkin) approach. Discontinuous Galerkin schemes for the same problem were analyzed in Yücel et al. [2013].

Variables & Notation

Unknowns

$$\begin{aligned} u &\in L^2(\Omega) && \text{control variable} \\ y &\in H^1(\Omega) && \text{state variable} \end{aligned}$$

Given Data

The given data is chosen in a way which admits an analytic solution. The diffusion parameter $\varepsilon > 0$ and the constant convection direction $\boldsymbol{c}$ can be freely chosen. In Collis and Heinkenschloss [2002] and Yücel et al. [2013], the values $\varepsilon = 10^{-2}$ and

$$\begin{aligned} \boldsymbol{c} &= (\cos\theta, \sin\theta)^\top \qquad \text{with } \theta = 45^\circ = \pi/4 \\ &= (1/2)\sqrt{2}\,(1,1)^\top \end{aligned}$$

were used.

$\Omega = (0,1)^2$	computational domain
Γ	its boundary
$x = (x_1, x_2)$	coordinate in Ω
$\eta(z) = z - \dfrac{\exp((z-1)/\varepsilon) - \exp(-1/\varepsilon)}{1-\exp(-1/\varepsilon)}$	boundary layer function
$\eta'(z) = 1 - \dfrac{1}{\varepsilon} \cdot \dfrac{\exp((z-1)/\varepsilon)}{1-\exp(-1/\varepsilon)}$	its first derivative
$\eta''(z) = \quad - \dfrac{1}{\varepsilon^2} \cdot \dfrac{\exp((z-1)/\varepsilon)}{1-\exp(-1/\varepsilon)}$	its second derivative
$\xi(z) = \quad \eta(1-z)$	boundary layer function
$\xi'(z) = -\eta'(1-z)$	its first derivative
$\xi''(z) = \quad \eta''(1-z)$	its second derivative
$f = -\varepsilon\, \eta''(x_1)\, \eta(x_2) - \varepsilon\, \eta(x_1)\, \eta''(x_2) + \boldsymbol{c} \cdot \begin{pmatrix} \eta'(x_1)\, \eta(x_2) \\ \eta(x_1)\, \eta'(x_2) \end{pmatrix} - \xi(x_1)\, \xi(x_2)$	right hand side
$y_d = -\varepsilon\, \xi''(x_1)\, \xi(x_2) - \varepsilon\, \xi(x_1)\, \xi''(x_2) - \boldsymbol{c} \cdot \begin{pmatrix} \xi'(x_1)\, \xi(x_2) \\ \xi(x_1)\, \xi'(x_2) \end{pmatrix} + \eta(x_1)\, \eta(x_2)$	desired state

Problem Description

$$\text{Minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}$$

$$\text{s.t.} \quad \begin{cases} -\varepsilon \triangle y + \boldsymbol{c} \cdot \nabla y = f + u & \text{in } \Omega \\ \hfill y = 0 & \text{on } \Gamma. \end{cases}$$

Optimality System

The following optimality system for the state $y \in H^1_0(\Omega)$, the control $u \in L^2(\Omega)$, the adjoint state $p \in H^1_0(\Omega)$ characterizes the unique minimizer.

$$\begin{aligned} -\varepsilon \triangle y + \boldsymbol{c} \cdot \nabla y &= f + u && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma \\ -\varepsilon \triangle p - \boldsymbol{c} \cdot \nabla p - (\operatorname{div} \boldsymbol{c})\, p &= -(y - y_d) && \text{in } \Omega \\ p &= 0 && \text{on } \Gamma \\ u - p &= 0 && \text{in } \Omega. \end{aligned}$$

Note that the coefficient $(\operatorname{div} \boldsymbol{c})$ in the adjoint equation vanishes for the given constant velocity field $\boldsymbol{c}$.

Supplementary Material

The optimal state, adjoint state, control and state constraint multiplier are known analytically:

$$\begin{aligned} y &= \eta(x_1)\,\eta(x_2) \\ p &= \xi(x_1)\,\xi(x_2) \\ u &= p. \end{aligned}$$

Note that the boundary layer for the state lies near the right and upper boundary (where $x_1 = 1$ or $x_2 = 1$), while the boundary layer for the adjoint state is located on the opposite parts of the boundary.

References

S. S. Collis and M. Heinkenschloss. Analysis of the streamline upwind/Petrov Galerkin method applied to the solution of optimal control problems. Technical Report CAAM TR02–01, Rice University, 2002. URL http://www.caam.rice.edu/~heinken/papers/supg_analysis.pdf.

H. Yücel, M. Heinkenschloss, and B. Karasözen. Distributed optimal control of diffusion-convection-reaction equations using discontinuous Galerkin methods. *Numerical Mathematics and Advanced Applications*, pages 389–397, 2013. doi: 10.1007/978-3-642-33134-3_42.