Introduction

Here we have a simple distributed optimal control problem of the Poisson equation with a potential term, and with pointwise bound constraints on the control. Problems of this type are treated extensively in [Tröltzsch, 2010, Chapter 2], and are sometimes refered to as the *mother problem* type. The present problem is special in the sense that the control acts in a distributed way on the entire domain Ω, and that the state is observed on the entire domain as well. Moreover, the non-trivial part of the solution is rotationally symmetric.

This problem and analytical solution appear in [Tröltzsch, 2010, Section 2.9.2].

Variables & Notation

Unknowns

$u \in L^2(\Omega)$	control variable
$y \in H^1(\Omega)$	state variable

Given Data

The given data is chosen in a way which admits an analytic solution. This solution is rotationally symmetric on a circle of radius $1/3$ strictly contained in Ω.

$\Omega = (0,1)^2$	computational domain
Γ	its boundary
$\widehat{x} = (0.5, 0.5)$	center of Ω
$y_\Omega = -\frac{142}{3} + 12\,\|x - \widehat{x}\|^2$	desired state
$e_\Omega = 1 - \operatorname{proj}_{[0,1]}\left(12\,\|x - \widehat{x}\|^2 - \frac{1}{3}\right)$	uncontrolled force
$e_\Gamma = -12$	boundary observation coefficient

Problem Description

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{2}\|y - y_\Omega\|^2_{L^2(\Omega)} + \int_\Gamma e_\Gamma\, y \,\mathrm{d}s + \frac{1}{2}\|u\|^2_{L^2(\Omega)} \\
\text{s.t.} \quad & \begin{cases} -\triangle y + y = u + e_\Omega & \text{in } \Omega \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \Gamma \end{cases} \\
\text{and} \quad & 0 \le u(x) \le 1 \quad \text{in } \Omega.
\end{aligned}
$$

Optimality System

The following optimality system for the state $y \in H_0^1(\Omega)$, the control $u \in L^2(\Omega)$ and the adjoint state $p \in H_0^1(\Omega)$, given in the strong form, characterizes the unique minimizer.

$$\begin{aligned} -\triangle y + y &= u + f && \text{in } \Omega \\ \frac{\partial y}{\partial n} &= 0 && \text{on } \Gamma \\ -\triangle p + p &= y - y_\Omega && \text{in } \Omega \\ \frac{\partial p}{\partial n} &= e_\Gamma && \text{on } \Gamma \\ u &= \operatorname{proj}_{[0,1]}(-p) && \text{in } \Omega \end{aligned}$$

Supplementary Material

The optimal state, adjoint state and control are known analytically:

$$\begin{aligned} &y = 1 \\ &p = -12\,\|x - \widehat{x}\|^2 + \frac{1}{3} \\ &u = \begin{cases} 1, & x \in \Omega_3 \\ 12\,\|x - \widehat{x}\|^2 - 1/3, & x \in \Omega_2 \\ 0, & x \in \Omega_1 \end{cases} \end{aligned}$$

where the subdomains are defined as follows:

$$\begin{aligned} \Omega_1 &= \{x \in \Omega : \|x - \widehat{x}\| < 1/6\} \\ \Omega_2 &= \{x \in \Omega : 1/6 \le \|x - \widehat{x}\| < 1/3\} \\ \Omega_3 &= \{x \in \Omega : 1/3 \le \|x - \widehat{x}\|\}. \end{aligned}$$

References

F. Tröltzsch. *Optimal Control of Partial Differential Equations*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, 2010.