Introduction

Here we have a simple boundary optimal control problem of the Poisson equation with pointwise box constraints on the control. The domain is polygonal, and it is the intersection of a square and a circular sector. The regularity of the optimal solution and consequently the approximation properties of numerical solutions depend on the angle ω of the circular sector.

This problem and the analytical example were published in Mateos and Rösch [2011].

Variables & Notation

Unknowns

$$
\begin{aligned}
u &\in L^2(\Gamma_\omega) && \text{control variable}\\
y &\in H^1(\Omega_\omega) && \text{state variable}
\end{aligned}
$$

Given Data

The given data is chosen in a way which admits an analytic solution. The domain Ω_ω and the solution depend on the angle $0 < \omega < 2\pi$. The most interesting cases arise when ω is the largest angle, i.e., in case $\omega \geq \pi/2$. The optimal control has the natural low regularity described by the singular exponent λ, which also depends on ω.

The description of the problem is most convenient when both cartesian coordinates (x_1, x_2) and polar coordinates (r, ϕ) are used interchangeably.

$$
\begin{aligned}
S_\omega &= \{(r\cos\phi, r\sin\phi) :\ r \in [0, \sqrt{2}),\ \phi \in (0, \omega)\} && \text{circular sector}\\
\Omega_\omega &= (-1,1)^2 \cap S_\omega && \text{computational domain}\\
\Gamma_\omega & && \text{its boundary}\\
\lambda &= \pi/\omega && \text{singular exponent}\\
y_d(r,\phi) &= -r^\lambda \cos(\lambda\,\phi) && \text{desired state (polar coordinates)}\\
g_1(x_1,x_2) &= -\frac{\partial}{\partial n} y_d(x_1,x_2) && \text{objective term}\\
g_2(x_1,x_2) &= -\operatorname{proj}_{[-0.5,0.5]}\big(y_d(x_1,x_2)\big) && \text{boundary term}
\end{aligned}
$$

The function g_1 can be computed by the formula

$$
g_1 = -\frac{\partial}{\partial n} y_d = -\nabla y_d \cdot n
$$

with n denoting the outer normal vector to Γ_ω, and

$$\nabla y_d = \begin{pmatrix} \frac{\partial y_d}{\partial x_1} \\ \frac{\partial y_d}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -\lambda\, r^{\lambda-1} \cos(\lambda\, \phi) \frac{x_1}{r} - \lambda\, r^{\lambda} \sin(\lambda\, \phi) \frac{x_2}{r^2} \\ -\lambda\, r^{\lambda-1} \cos(\lambda\, \phi) \frac{x_2}{r} + \lambda\, r^{\lambda} \sin(\lambda\, \phi) \frac{x_1}{r^2} \end{pmatrix}.$$

Note that g_1 vanishes at the part of Γ_ω that coincides with the boundary of the circular sector.

Problem Description

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_\omega)} + \int_{\Gamma_\omega} g_1\, y \,\mathrm{d}s + \frac{1}{2}\|u\|^2_{L^2(\Gamma_\omega)} \\ \text{s.t.} \quad & \begin{cases} -\triangle y + y = 0 & \text{in } \Omega_\omega \\ \dfrac{\partial y}{\partial n} = u + g_2 & \text{on } \Gamma_\omega \end{cases} \\ \text{and} \quad & -0.5 \le u(x_1, x_2) \le 0.5 \quad \text{on } \Gamma_\omega. \end{aligned}$$

Optimality System

The following optimality system for the state $y \in H^1_0(\Omega_\omega)$, the control $u \in L^2(\Gamma_\omega)$ and the adjoint state $p \in H^1_0(\Omega_\omega)$, given in the strong form, characterizes the unique minimizer.

$$\begin{aligned} -\triangle y + y &= 0 && \text{in } \Omega_\omega \\ \frac{\partial y}{\partial n} &= u + g_2 && \text{on } \Gamma_\omega \\ -\triangle p + p &= y - y_d && \text{in } \Omega_\omega \\ \frac{\partial p}{\partial n} &= g_1 && \text{on } \Gamma_\omega \\ u &= \operatorname{proj}_{[-0.5,0.5]}(-p|_{\Gamma_\omega}) && \text{on } \Gamma_\omega \end{aligned}$$

Supplementary Material

The optimal state, adjoint state and control are known analytically:

$$\begin{aligned} y &= 0 && \text{in } \Omega_\omega \\ p &= -y_d && \text{in } \Omega_\omega \\ u &= -g_2 && \text{on } \Gamma_\omega \end{aligned}$$

References

M. Mateos and A. Rösch. On saturation effects in the Neumann boundary control of elliptic optimal control problems. *Computational Optimization and Applications*, 49 (2):359–378, 2011. doi: 10.1007/s10589-009-9299-5.